Simple linear regression analysis

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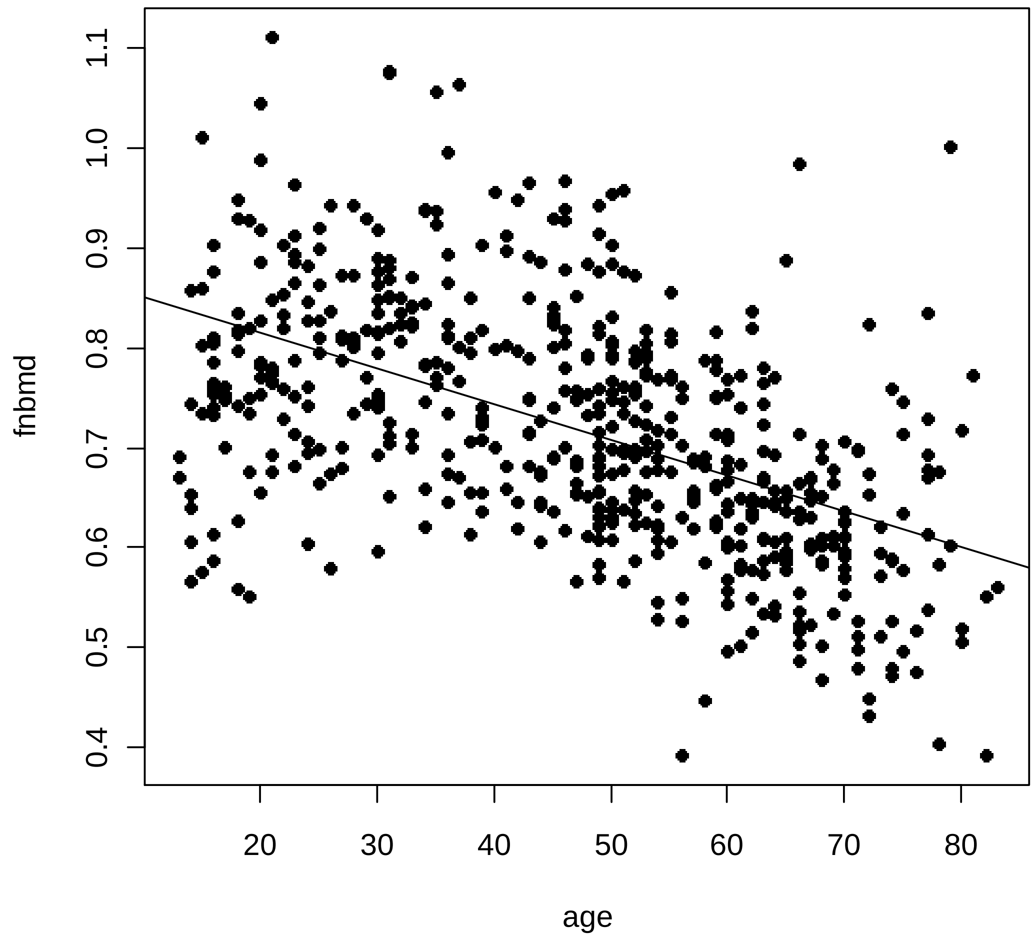
What we are going to learn …

- Examples
- Purposes of linear regression analysis
- Questions of interest
- Model parameters
- R analysis
- Interpretation

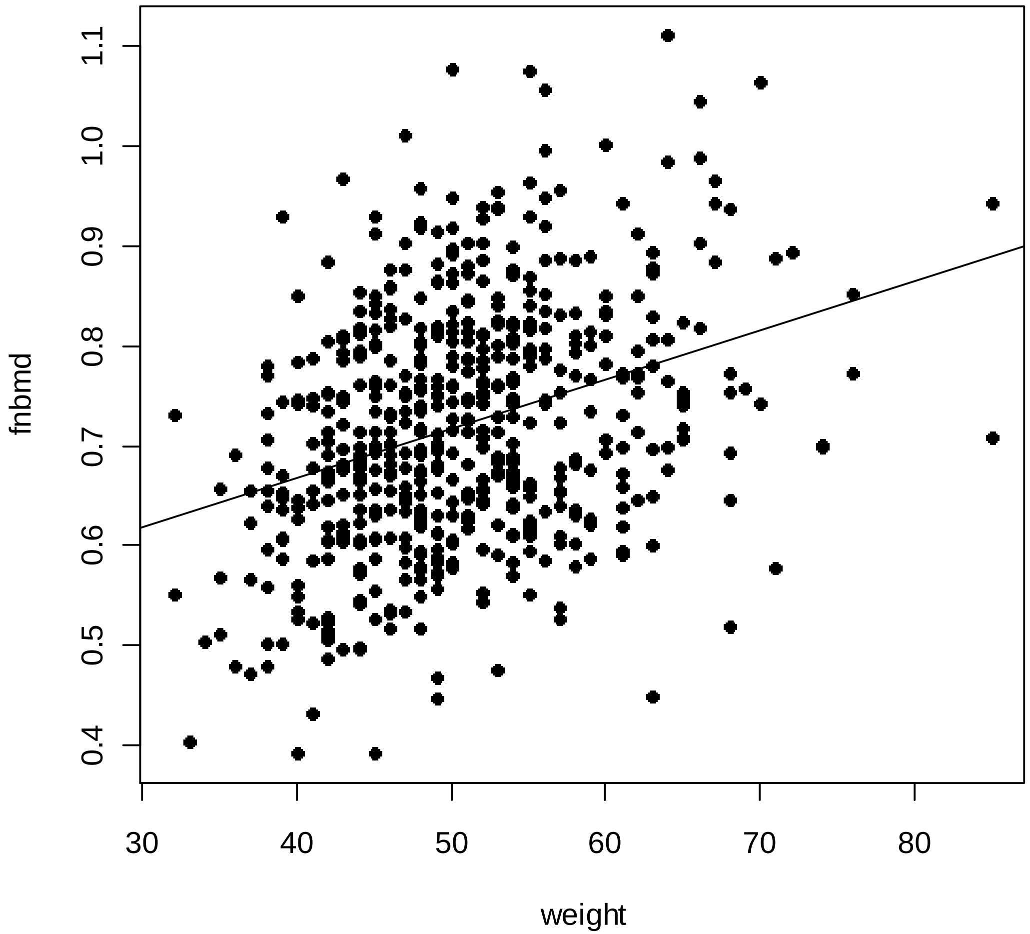
Femoral neck bone density and age

```
women = subset(vd, sex==2)
plot(fnbmd ~ age, pch=16)
abline(lm(fnbmd ~ age))
```



Weight and femoral neck bone density

```
plot(fnbmd ~ weight, pch=16)
abline(lm(fnbmd ~ weight))
```



Correlation analysis

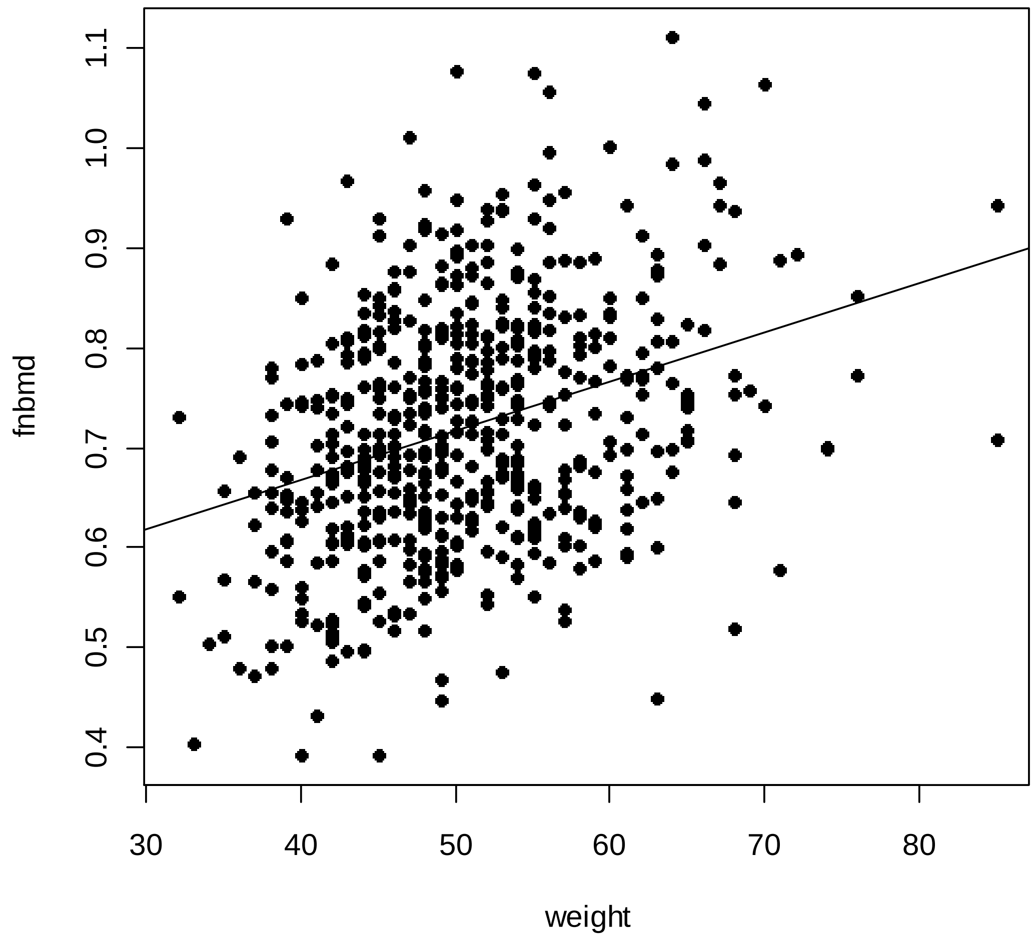
- Assessment of a relationship
- The coefficient of correlation: a measure of the relationship
- We want to know more …
  - The magnitude of effect of a *predictor* variable on the *outcome*
  - Prediction of outcome by using the predictor variable(s)

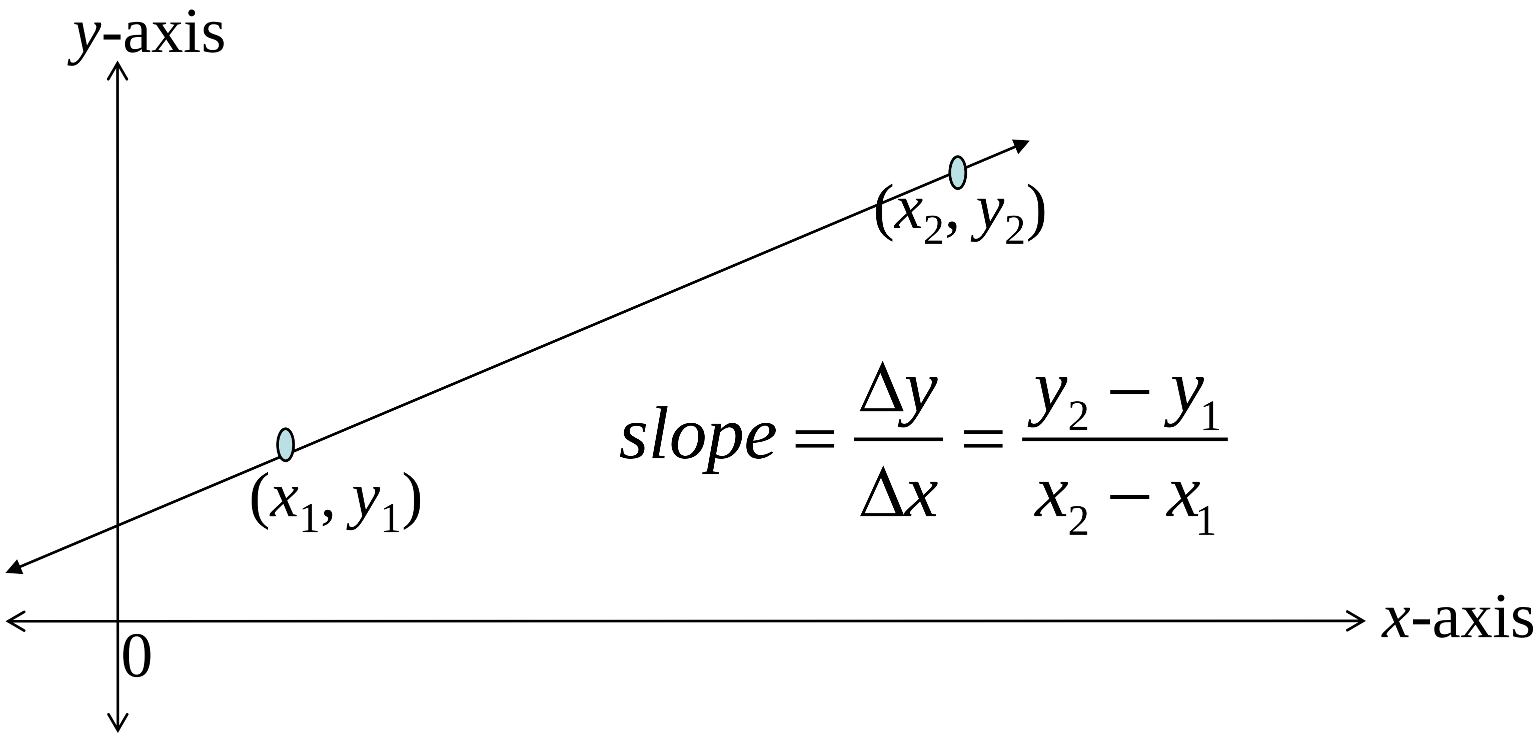
Our interests

- Finding a statistical model that decribes the relationship between age, weight, and BMD
- Adjustment of effect
- Prediction

Linear regression model

Weight and femoral neck bone density





$$slope = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

We can also describe the line in terms of a slope and an intercept

- **The slope the change in the *y*-value for a unit change in the *x*-value. In this simple situation we can think of this as the change in the height of the line as we progress along the *x*-axis**
- **The intercept is the height of the line when *x* = 0**

Linear regression: model

- *Y* : random variable representing a **response**
- *X* : random variable representing a **predictor** variable (predictor, risk factor)
  - Both *Y* and *X* can be a categorical variable (e.g., yes / no) or a continuous variable (e.g., age).
  - If *Y* is categorical, the model is a **logistic regression** model; if *Y* is continuous, a **simple linear regression** model.

- **Model**

$$Y = \alpha + \beta X + \varepsilon$$

α : intercept

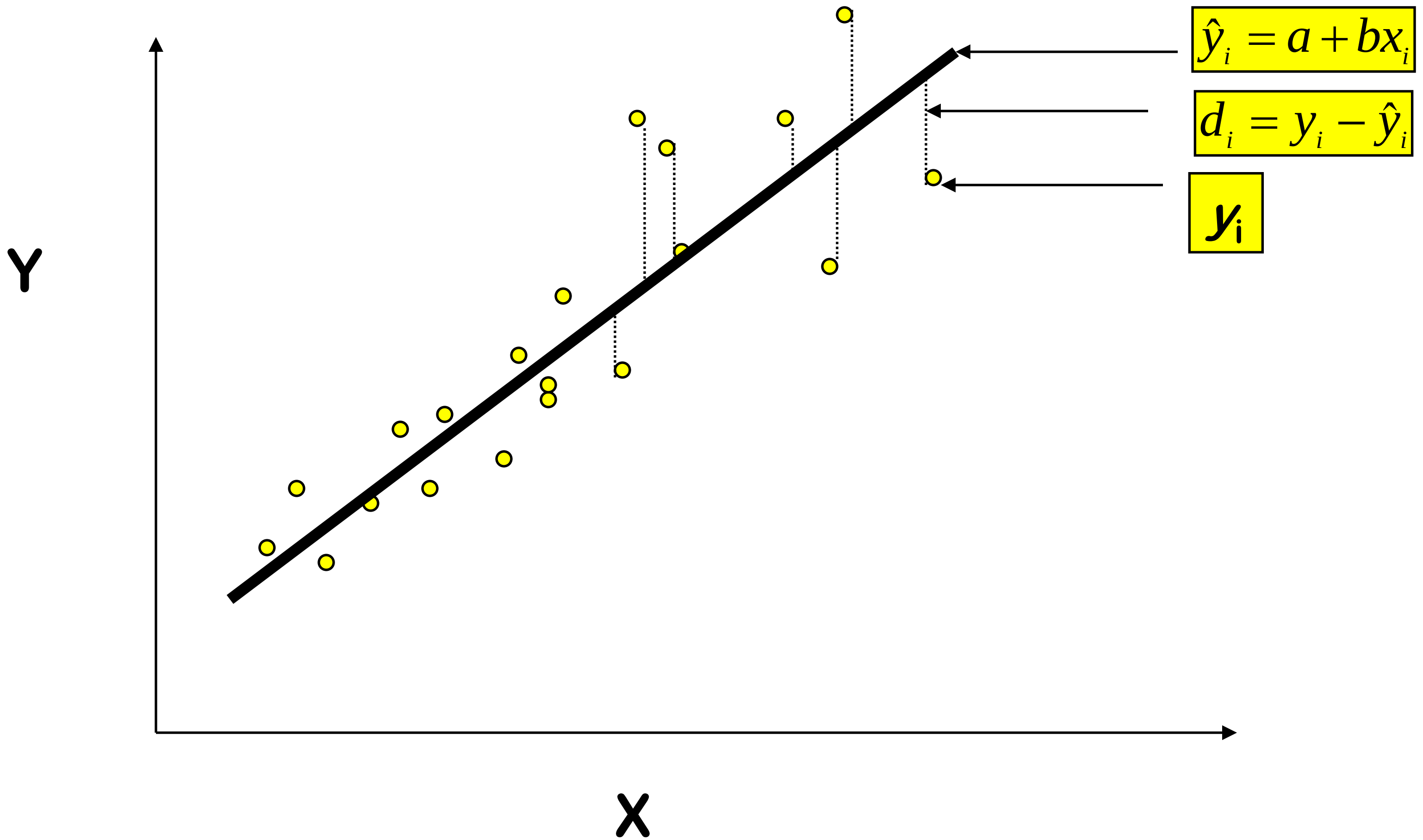
β : slope / gradient

ε : random error (variation between subjects in y even if *x* is constant, e.g., variation in cholesterol for patients of the same age.)

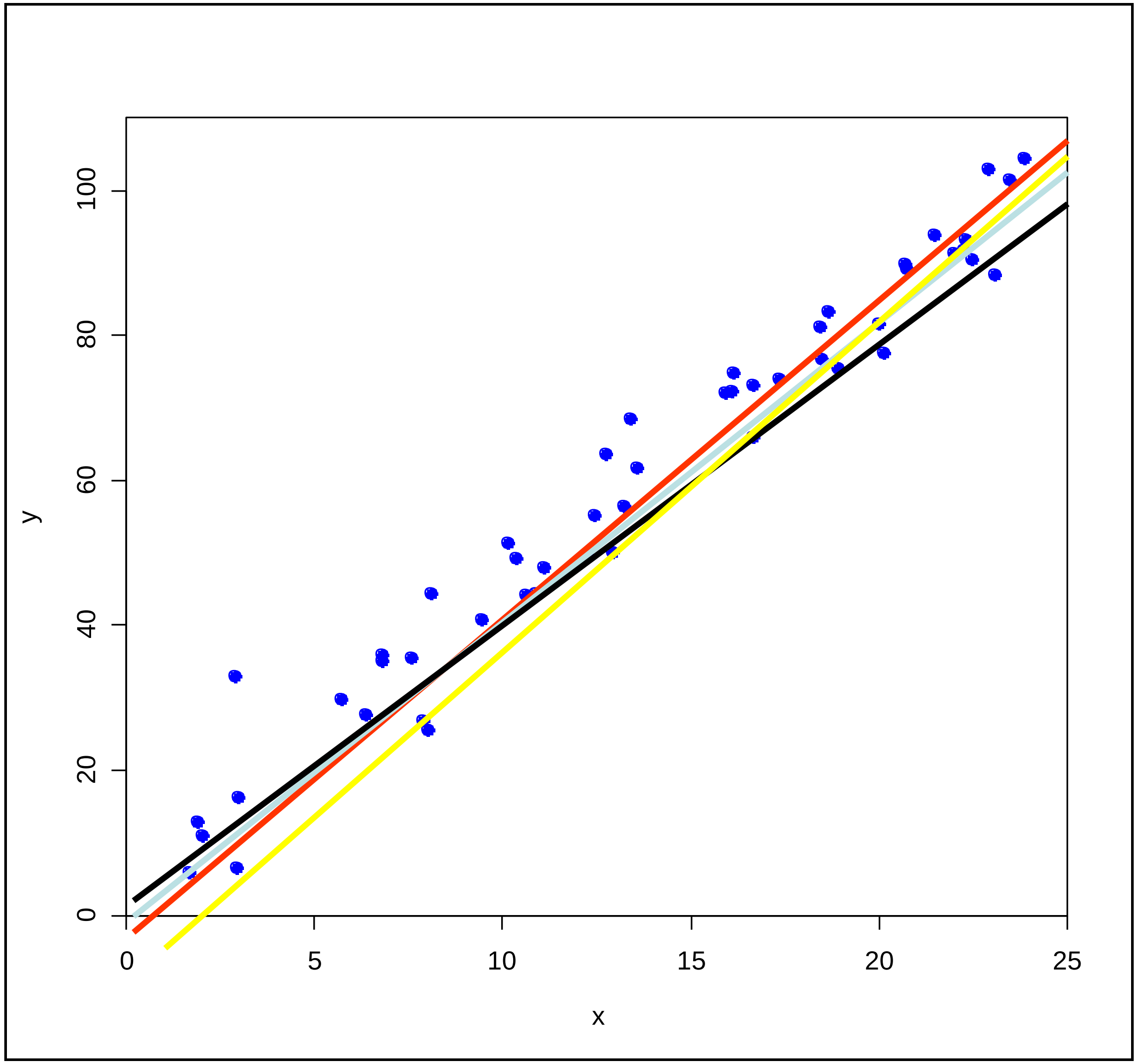
Linear regression: assumptions

- The relationship is linear *in terms of the parameter*;
- *X* is measured without error;
- The values of *Y* are independently from each other (e.g., Y_1 is not correlated with Y_2) ;
- The random error term (ε) is **normally** distributed with mean 0 and **constant** variance.

Criteria of estimation



The goal of least square estimator (LSE) is to find a and b such that the sum of d^2 is minimal.



- We could try fitting a line “by eye”
- But everyone’s best guess would probably be different
- We want consistency

Estimating parameters by R

- **Our interest: relationship between BMD and weight**
- **Model:**

$$BMD = a + b*weight + e$$

- **We want to estimate *a* and *b***
- **R language**

```
lm(bmd ~ weight)
```

R analysis

```
> m1 = lm(fnbmd ~ weight)
> summary(m1)
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.4699822  0.0310144   15.15  < 2e-16 ***
weight      0.0049416  0.0006041    8.18 1.95e-15 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.1152 on 556 degrees of freedom
Multiple R-squared: 0.1074,     Adjusted R-squared: 0.1058
F-statistic:  66.9 on 1 and 556 DF,  p-value: 1.945e-15
```

Interpretation of outputs

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.4699822  0.0310144   15.15  < 2e-16 ***
weight      0.0049416  0.0006041    8.18 1.95e-15 ***
```

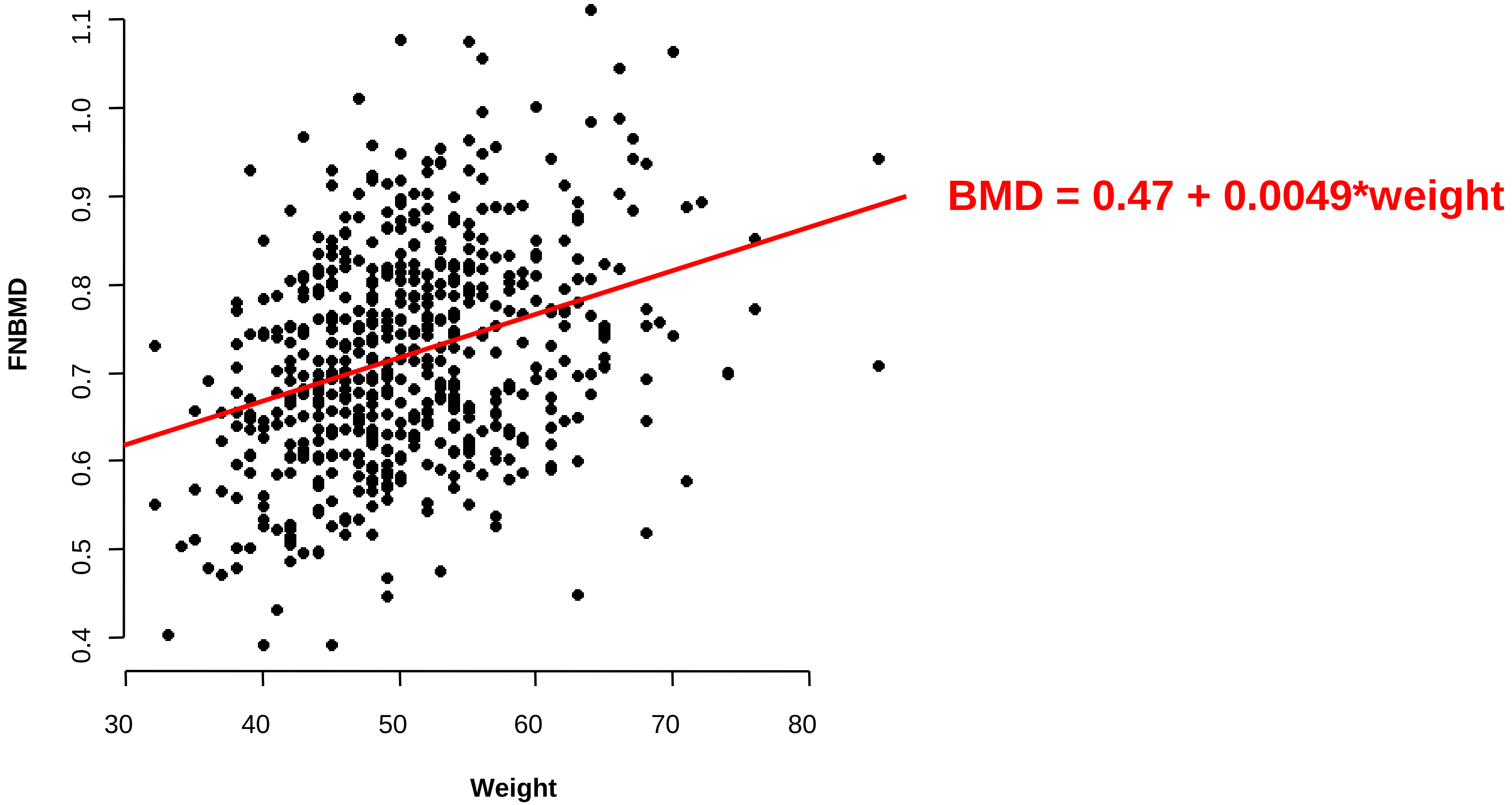
- Remember our model:

$$BMD = a + b*weight$$

- Our equation:

$$BMD = 0.47 + 0.0049*weight$$

- Interpretation: **1 kg increase in weight was associated with a 0.0049 g/cm^2 increase in BMD. The association is statistically significant (P < 0.0001)**


BMD = 0.47 + 0.0049*weight
FNBMD
Weight
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
30
40
50
60
70
80

Analysis of variance

- BMD = a + b*weight + e
- Observed variation = model + random

  "Variation" = sum of squares

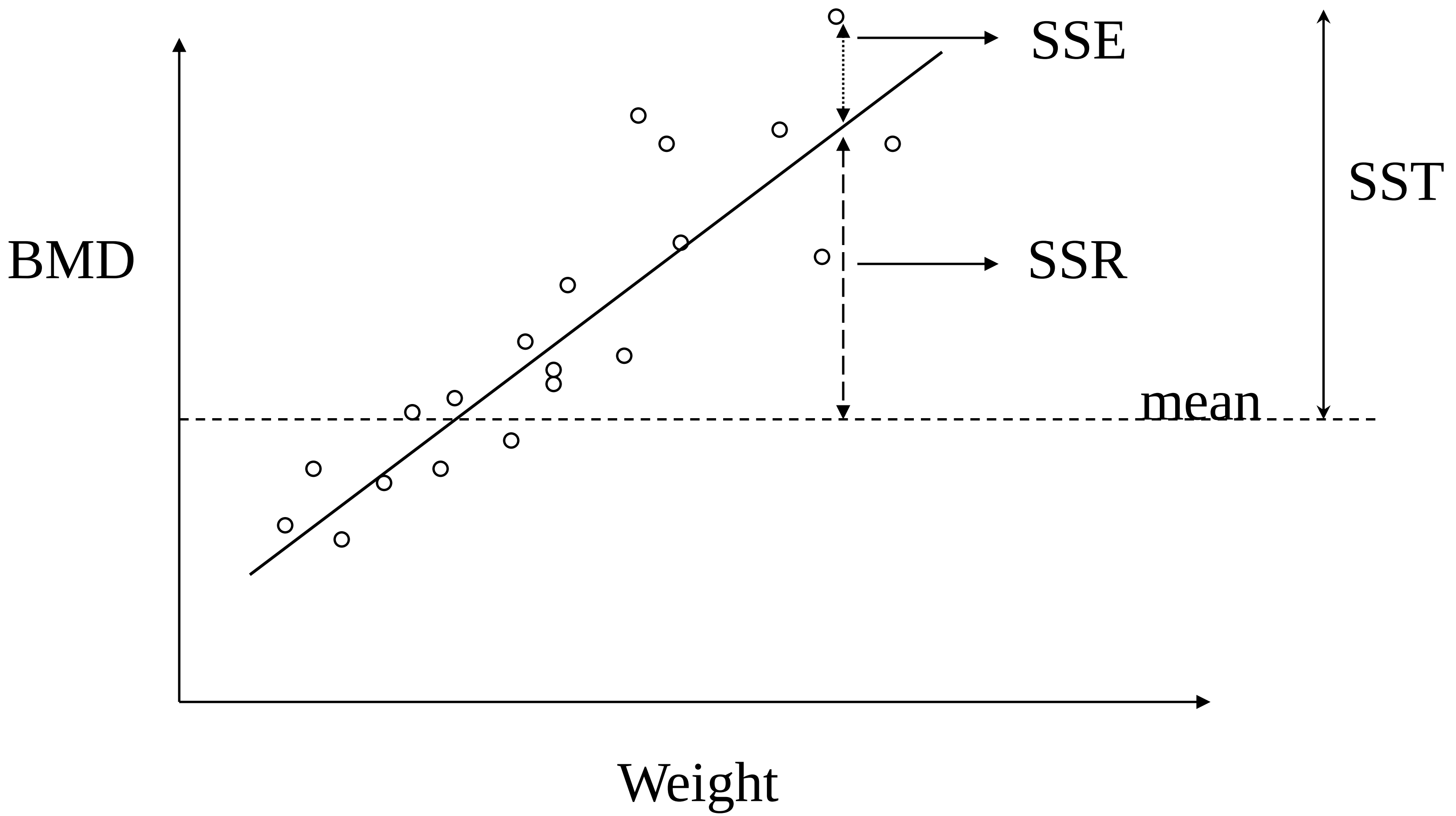
- SST = total sum of squares

  SSR = sum of squares due to the regresson model

  SSE = sum of squares due to random component

- SST = SSR + SSE
- R^2 = SSR / SST

Partitioning of variations: geometry



SST = SSR + SSE

Partitioning of variation by R

```
> m1 = lm(fnbmd ~ weight)
> anova(m1)
```

```
Analysis of Variance Table
Response: fnbmd
           Df Sum Sq Mean Sq F value    Pr(>F)
weight      1 0.8883 0.88829  66.905 1.945e-15 ***
Residuals 556 7.3819 0.01328
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- Total SS = 0.8883 + 7.3819 = 8.2702
- R2 = 0.8883 / 8.2702 = 0.107

Interpretation of outputs

```
Residual standard error: 0.1152 on 556 degrees of freedom
Multiple R-squared: 0.1074,     Adjusted R-squared: 0.1058
F-statistic:  66.9 on 1 and 556 DF,  p-value: 1.945e-15
```

- $R^2 = 0.107$
- Interpretation: **Approximately 11% of BMD variance could be accounted for by body weight**

Variance of BMD after adjusting for weight

```
> m1 = lm(fnbmd ~ weight)
> anove(m1)
```

```
Analysis of Variance Table
Response: fnbmd
           Df Sum Sq Mean Sq F value    Pr(>F)
weight      1 0.8883 0.88829  66.905 1.945e-15 ***
Residuals 556 7.3819 0.01328
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- Mean square (MS) = sum of squares / (degrees of freedom)
- MS(residuals) = 7.3819 / 556 = 0.01328

⇒ Variance of BMD after adjusting for weight is 0.01328

(variance of BMD before the adjustment: 0.01485

Prediction of BMD by weight

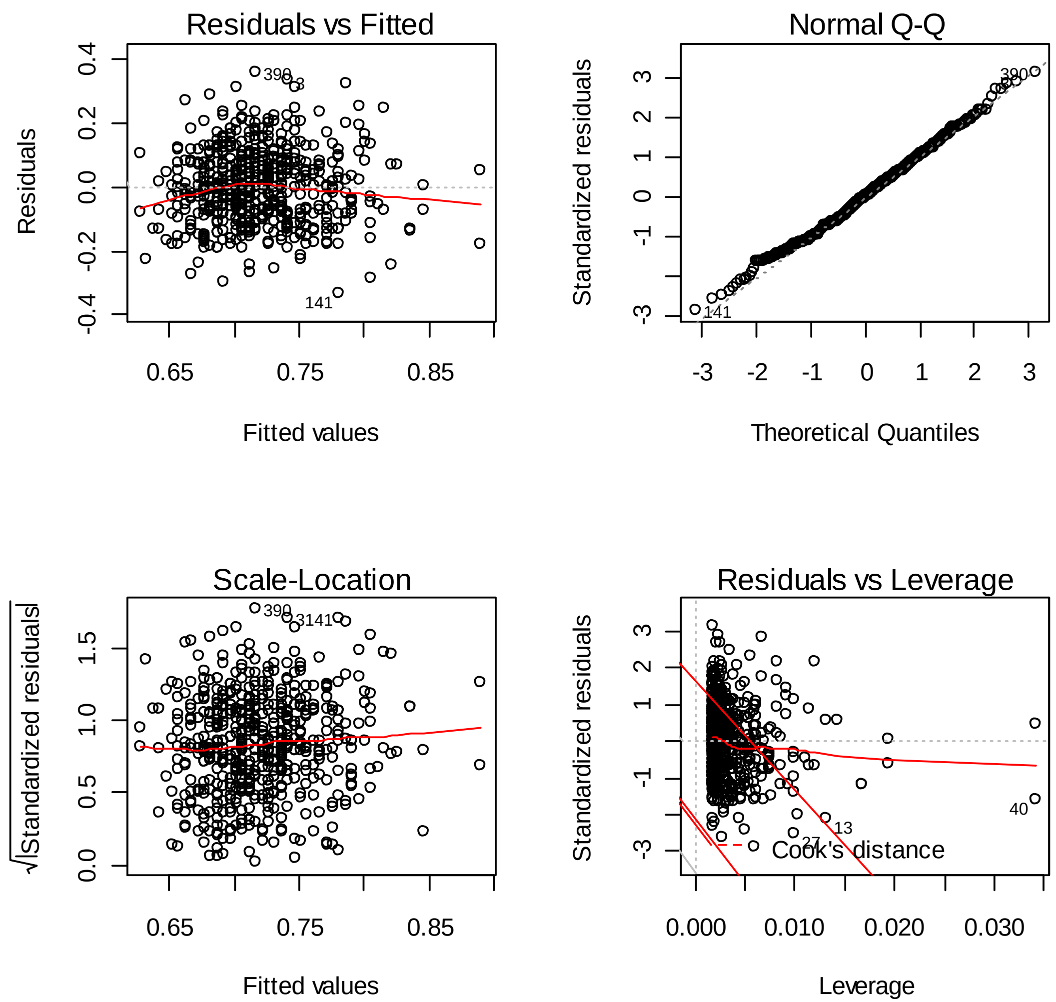
- The model: BMD = 0.47 + 0.0049*weight
- Without the knowledge of weight, the mean BMD is 0.72 g/cm^2
- With knowledge of weight, we know that BMD is dependent on weight
- Weight = 50 kg, BMD = 0.47 + 0.0049*50 = 0.72 g/cm^2

  Weight = 40 kg, BMD = 0.47 + 0.0049*40 = 0.67 g/cm^2

  Weight = 60 kg, BMD = 0.47 + 0.0049*60 = 0.76 g/cm^2

Checking model assumptions

```
par(mfrow=c(2,2))
plot(m1)
```



Be careful! Anscrombe's data

Frank Anscombe devised 4 sets of X-Y pairs

x	y1	y2	y3	x4	y4
10	8.04	9.14	7.46	8	6.58
8	6.95	8.14	6.77	8	5.76
13	7.58	8.74	12.74	8	7.71
9	8.81	8.77	7.11	8	8.84
11	8.33	9.26	7.81	8	8.47
14	9.96	8.10	8.84	8	7.04
6	7.24	6.13	6.08	8	5.25
4	4.26	3.10	5.39	19	12.50
12	10.84	9.13	8.15	8	5.56
7	4.82	7.26	6.42	8	7.91
5	5.68	4.74	5.73	8	6.89

Mean and SD of Anscombe's data

Data Set	N	X		Y	
		Mean	SD	Mean	SD
1	11	7.50	2.03	9.00	3.32
2	11	7.50	2.03	9.00	3.32
3	11	7.50	2.03	9.00	3.32
4	11	7.50	2.03	9.00	3.32

Correlation between X and Y: Anscombe's data

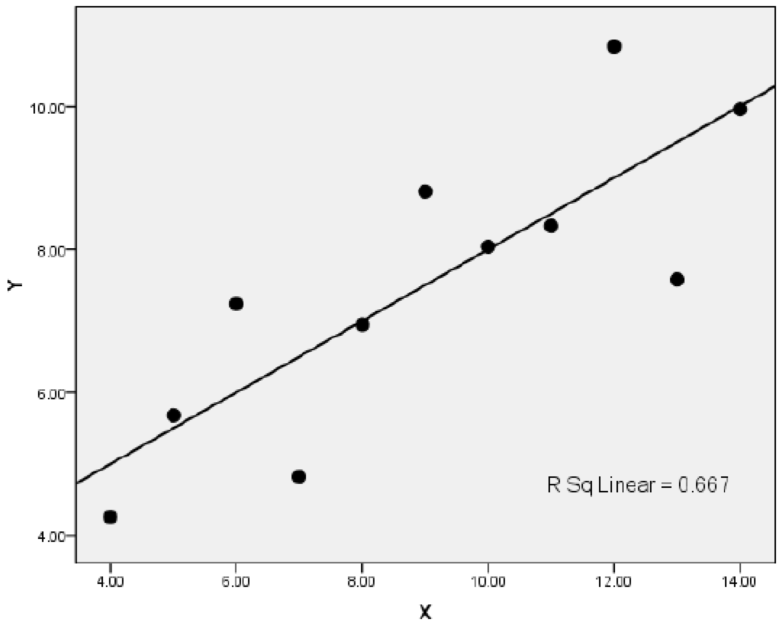
Data Set	Pearson r	R-squared	Adj. R-sq	SE
1	0.82	0.67	0.63	1.24
2	0.82	0.67	0.63	1.24
3	0.82	0.67	0.63	1.24
4	0.82	0.67	0.63	1.24

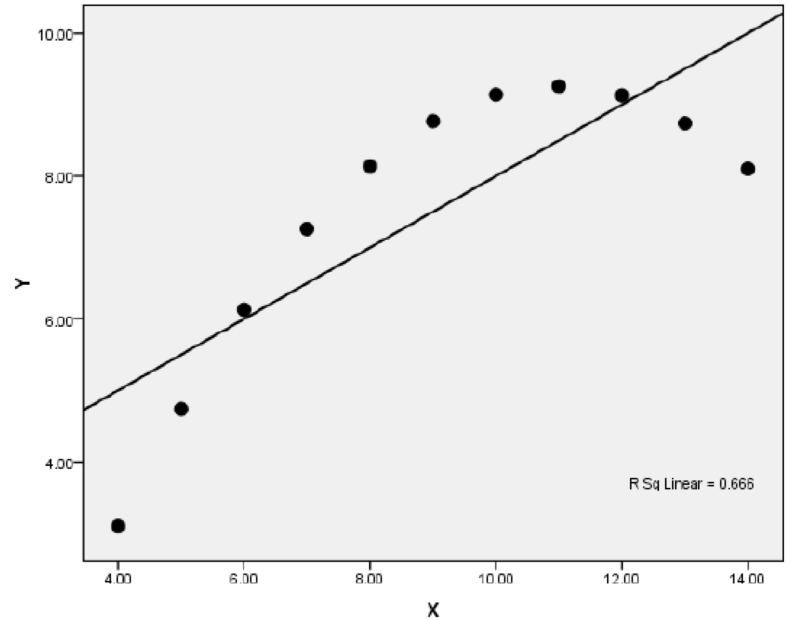
Regression analysis: Anscombe's data

Data Set		B	SE	t	p	95% CI Lower	95% CI Upper
1	Constant	3.00	1.124	2.67	0.026	0.459	5.544
	X	0.50	0.118	4.24	0.002	0.233	0.766
2	Constant	3.00	1.124	2.67	0.026	0.459	5.546
	X	0.50	0.118	4.24	0.002	0.233	0.766
3	Constant	3.00	1.125	2.67	0.026	0.455	5.547
	X	0.50	0.118	4.24	0.002	0.233	0.767
4	Constant	3.00	1.125	2.67	0.026	0.456	5.544
	X	0.50	0.118	4.24	0.002	0.233	0.767

For all 4 models, $Y' = 0.5(X) + 3$

But …


R Sq Linear = 0.667
Y
X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Y
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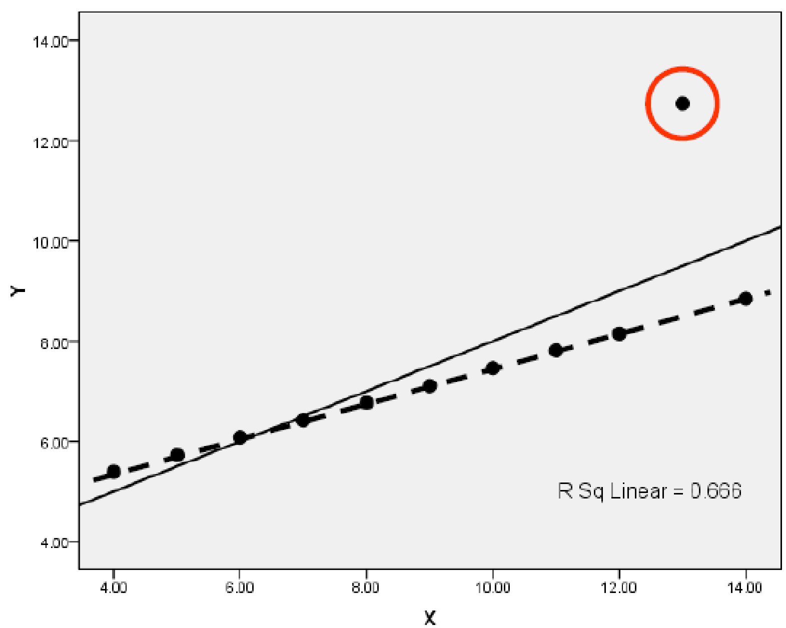
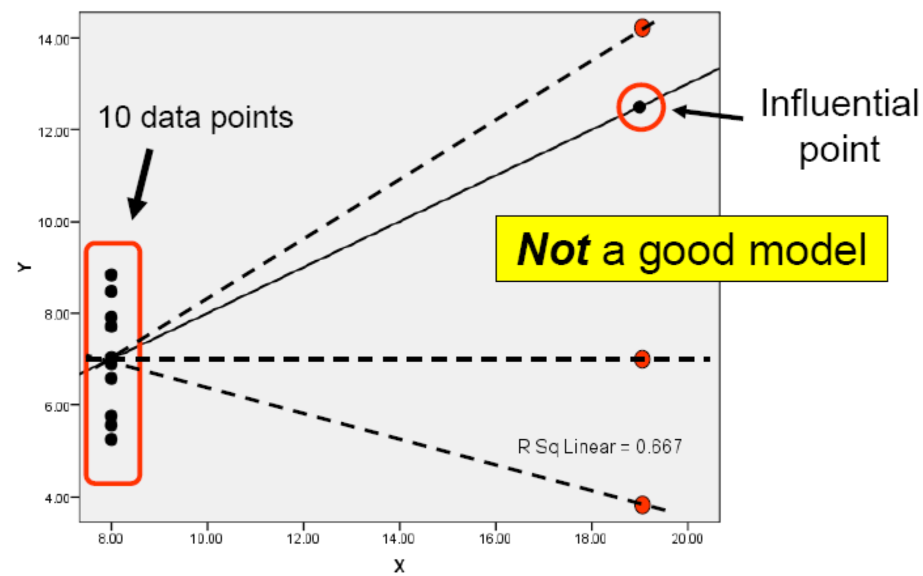

R Sq Linear = 0.666
Y
X


10 data points
Influential point
Not a good model
R Sq Linear = 0.667
Y
X

Summary

- Simple linear regression model is used for
  - Understanding the effect of a risk factor or determinant on an outcome variable
  - Predicting an outcome variable
- It’s appropriate when the functional relationship is linear
- Always check assumptions!